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On optimal orientation of cycle vertex multiplications

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Abstract

For a bridgeless connected graph G, let $\mathscr{D}(G)$ be the family of its strong orientations; and for any $D \in \mathscr{D}(G)$, we denote by $d(D)$ its diameter. The orientation number $\vec{d}(G)$ of G is defined by $\vec{d}(G) = \min\{d(D) | D \in \mathscr{D}(G)\}$. For a connected graph G of order n and for any sequence of n positive integers (s_i), let $G(s_1, s_2, \ldots, s_n)$ denote the graph with vertex set V^* and edge set E^* such that $V^* = \bigcup_{i=1}^{n} V_i$, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, $i = 1, 2, \ldots, n$, and for any two distinct vertices x, y in V^*, $xy \in E^*$ if and only if $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. We call the graph $G(s_1, s_2, \ldots, s_n)$ a G vertex multiplication. In this paper, we determine the orientation numbers of various cycle vertex multiplications.

Keywords: Strong orientation; Orientation number; Cycle; Vertex multiplication

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity* $e(v)$ of v is defined as $e(v) = \max\{d(v, x) | x \in V(G)\}$, where $d(v, x)$ denotes the distance from v to x. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e(v) | v \in V(G)\}$. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. For $v \in E(D)$, the notions $e(v)$ and $d(D)$ are similarly defined.

An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if every two vertices in D are mutually reachable

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in D. An edge e in a connected graph is a bridge if $G-e$ is disconnected. Robbins' celebrated one-way street theorem [14] states that a *connected graph G has a strong orientation if and only if no edge of G is a bridge.* Efficient algorithms for finding a strong orientation for a bridgeless graph can be found in Roberts [15], Boesch and Tindell [1] and Chung et al. [2]. Boesch and Tindell [1] extended Robbins' result to mixed graphs where edges could be directed or undirected. Chung et al. [2] provided a linear-time algorithm for testing whether a mixed graph has a strong orientation and finding one if it does. As another possible way of extending Robbins' theorem, consider further the notion $\rho(G)$ given below. Given a connected graph G containing no bridges, let $\mathscr{D}(G)$ be the family of its strong orientations. Define

$$\rho(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\} - d(G).$$

The first term on the right-hand side of the above equality is essential. We may write

$$\overrightarrow{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$$

and call it the *orientation number* of G. The problem of evaluating $\overrightarrow{d}(G)$ for an arbitrary connected graph G is very difficult. As a matter of fact, Chvátal and Thomassen [3] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

On the other hand, the parameter $\overrightarrow{d}(G)$ has been studied in various classes of graphs, including complete bipartite graphs [13,1,16,4] and complete n-partite graphs [13,4–6,8,9]. Most recently, Koh and Tay [10] extended the results on the complete n-partite graphs by introducing a new family of graphs based on a given connected graph as follows. Let G be a given connected graph of order n with vertex set $V(G)=\{v_1, v_2, \dots, v_n\}$. For any sequence of n positive integers (s_i), let $G(s_1, s_2, \dots, s_n)$ denote the graph with vertex set V^* and edge set E^* such that $V^* = \bigcup_{i=1}^{n} V_i$, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, $i = 1, 2, \dots, n$, and for any two distinct vertices x, y in V^*, $xy \in E^*$ if and only if $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. We call the graph $G(s_1, s_2, \dots, s_n)$ a *G vertex multiplication*. Thus when G is the complete graph K_n of order n, the graph $G(s_1, s_2, \dots, s_n)$ is a complete n-partite graph. For $s = 1, 2, \dots$, we shall denote $G(s, s, \dots, s)$ simply by $G^{(s)}$.

The fundamental result obtained by Koh and Tay in [10] is presented below.

Theorem A. *Let G be a connected graph of order $n \geqslant 3$. Given $s_i \geqslant 2$ for each $i=1, 2, \dots, n$, $d(G) \leqslant \overrightarrow{d}(G(s_1, s_2, \dots, s_n)) \leqslant d(G) + 2$.*

We may assume throughout this paper that $s_i \geqslant 2$ for each $i = 1, 2, \dots, n$. As a result of Theorem A, all graphs of the form $G(s_1, s_2, \dots, s_n)$ are divided into the following three classes: $\mathscr{C}_i = \{G(s_1, s_2, \dots, s_n) \mid \overrightarrow{d}(G(s_1, s_2, \dots, s_n)) = d(G) + i\}$, $i = 0, 1, 2$. When G is the cycle C_n of order n, the following results in [10] provide the motivation for this paper.

Theorem B. *Let G be a connected graph of order $n \geqslant 3$. If $d(G) \geqslant 4$ and $s_i \geqslant 4$ for each $i = 1, 2, \dots, n$, then $G(s_1, s_2, \dots, s_n) \in \mathscr{C}_0$.*

Theorem C. *For $n \geqslant 6$, $C_n(s_1, s_2, \dots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.*

Our objective in this paper is to investigate the following problem: for what sequence of integers (s_i), will $C_n(s_1, s_2, \dots, s_n)$ belong to $\mathscr{C}_0$? The following result will be established.

Theorem. (a) $C_n(s_1, s_2, \dots, s_n) \in \mathscr{C}_0$ *for all* $n \geqslant 10$ *and* $s_i \geqslant 3$ *for each* $i = 1, 2, \dots, n$;
(b) $C_n^{(3)} \in \mathscr{C}_1$ *for* $6 \leqslant n \leqslant 9$;
(c) $C_n^{(4)} \in \mathscr{C}_0$ *for* $n = 6, 7$.
(*Note that when* $n \geqslant 8$, $C_n^{(4)} \in \mathscr{C}_0$ *by Theorem* B.)

2. Terminology and a useful lemma

Let $V(C_n) = \{1, 2, \dots, n\}$. We shall write, for $i = 1, 2, \dots, n$, $V_i = \{(p, i) | 1 \leqslant p \leqslant s_i\}$ and call (p, i) the pth vertex in V_i. Thus two vertices (p, i) and (q, j) in $C_n(s_1, s_2, \dots, s_n)$ are adjacent if and only if $|i - j| = 1$ or $n - 1$. Let V^* be the vertex set of $C_n(s_1, s_2, \dots, s_n)$, $F \in \mathscr{D}(C_n(s_1, s_2, \dots, s_n))$. For $x, y \in V^*$, we write $x \to y$ or $y \leftarrow x$ if x is adjacent to y in F. Also, for $A, B \subseteq V^*$ with $A \cap B = \emptyset$, we write $A \to B$ or $B \leftarrow A$ if $x \to y$ in F for all $x \in A$ and for all $y \in B$. When $A = \{x\}$, we shall write $x \to B$ or $B \leftarrow x$ for $A \to B$.

For any subdigraph A of F. The *out-set* and *in-set* of a vertex (p, i) in A are defined, respectively, as

$$\mathrm{O}_A((p, i)) = \{(q, j) \in V^* \mid (p, i) \to (q, j) \text{ in } A\},$$

and

$$I_A((p, i)) = \{(q, j) \in V^* \mid (q, j) \to (p, i) \text{ in } A\}.$$

The eccentricity of (p, i) in A is denoted by $e_A((p, i))$. The subscript A is omitted if $A = F$.

For any digraph D, the *converse* of D, denoted by $\bar{D}$ is the digraph obtained when all the arcs in D are reversed. That is, $V(D) = V(\bar{D})$ and $xy \in E(D)$ if and only if $yx \in E(\bar{D})$.

The following result, obtained by Koh and Tay [10], will be found useful.

Lemma D. *Let* t_i, s_i *be positive integers such that* $t_i \leqslant s_i$ *for* $1 \leqslant i \leqslant n$. *If the graph* $G(t_1, t_2, \dots, t_n)$ *admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then* $\overrightarrow{d}(G(s_1, s_2, \dots, s_n)) \leqslant \max\{m, d(F)\}$.

3. Proof of part (a) in the main theorem

In this section, we shall establish result (a) as stated in our main theorem. We shall first prove that $C_n^{(3)} \in C_0$ for $n \geqslant 10$. This shall be accomplished separately by two propositions, depending on the parity of n.

Proposition 1. *When* $n \geqslant 10$ *and n is even, then* $C_n^{(3)} \in \mathscr{C}_0$.

Proof. Let $F \in \mathscr{D}(C_n^{(3)})$ be defined as follows. For $1 \leqslant i \leqslant n$,

(i) if i is odd, $(1, i) \to \{(1, i+1), (2, i+1)\}$, $(2, i) \to \{(2, i+1), (3, i+1)\}$, $(3, i) \to (3, i+1)$;

(ii) if i is even, $(1, i) \to (2, i+1)$, $(2, i) \to \{(1, i+1), (3, i+1)\}$, $(3, i) \to (2, i+1)$;
(iii) for all (s, t), (p, q), where $1 \leqslant s, p \leqslant 3$ and $1 \leqslant t, q \leqslant n$, if $(s, t) \nrightarrow (p, q)$ by (i) and (ii) above, then let $(p, q) \to (s, t)$.

It is clear that when $|i - j| \leqslant 1$, $d((p, i), (q, j)) \leqslant 3$ for all $1 \leqslant p, q \leqslant 3$. We shall show that $d(F) \leqslant n/2$. By symmetry, it suffices to show that $e((i, j)) \leqslant n/2$ for $i = 1, 2, 3$ and $j = 1, 2$.

Consider $(1, 1)$ and (j, q) for $3 \leqslant q \leqslant (n/2) + 1$. If there is a path from $(1, 1)$ to (j, q) for all j which is of length $q - 1$, then it is clear that there is path of length q from $(1, 1)$ to $(j, q+1)$ for all j.

Since $(1, 1) \to (2, 2) \to \{(1, 3), (3, 3)\}$ and $(1, 1) \to (1, 2) \to (2, 3)$ are paths of length 2 from $(1, 1)$ to $(j, 3)$ for all j, there are paths of length $q - 1 (\leqslant n/2)$ from $(1, 1)$ to (j, q) for all j and $3 \leqslant q \leqslant (n/2) + 1$.

Likewise, for $(n/2) + 2 \leqslant q \leqslant n - 1$, if there is a path of length $n - q + 1$ from $(1, 1)$ to (j, q) for all j, then there is a path of length $n - q + 2$ from $(1, 1)$ to $(j, q - 1)$ for all j.

Since $(1, 1) \to (1, n) \to \{(2, n-1), (3, n-1)\}$ and $(1, 1) \to (3, n) \to (1, n-1)$ are paths of length 2 from $(1, 1)$ to $(j, n-1)$ for all j, there are paths of length $n - q + 2 (\leqslant n/2)$ from $(1, 1)$ to (j, q) for all j and $(n/2) + 2 \leqslant q \leqslant n - 1$.

Consider $(2, 1)$ and (j, q) for $3 \leqslant q \leqslant (n/2) + 1$. Since $(2, 1) \to (2, 2) \to \{(1, 3), (3, 3)\}$ and $(2, 1) \to (3, 2) \to (2, 3)$ are paths of length 2 from $(2, 1)$ to $(j, 3)$ for all j, there are paths of length not exceeding $n/2$ from $(2, 1)$ to (j, q) for all j and $3 \leqslant q \leqslant (n/2) + 1$.

For $(n/2) + 2 \leqslant q \leqslant n - 3$, note that the following paths in F from $(2, 1)$ to $(j, n-3)$ for all j are of length 4:

$$(2, 1) \to (2, n) \to (3, n-1) \to (3, n-2) \to (1, n-3);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (1, n-2) \to (2, n-3);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (1, n-2) \to (3, n-3).$$

Thus there are paths of length not exceeding $n/2$ from $(2, 1)$ to (j, q) for all j and $(n/2) + 2 \leqslant q \leqslant n - 3$. For $q = n - 1, n - 2$, the following paths from $(2, 1)$ to (j, q) are of length not exceeding $n/2$:

$$(2, 1) \to (2, n) \to (3, n-1) \to (3, n) \to (1, n-1);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (3, n-2) \to (2, n-1);$$

$$(2, 1) \to (2, n) \to (3, n-1);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (1, n-2);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (1, n-2) \to (2, n-1) \to (2, n-2);$$

$$(2, 1) \to (2, n) \to (3, n-1) \to (3, n-2).$$

Consider $(3, 1)$ and (j, q) for $5 \leqslant q \leqslant (n/2) + 1$. Note that the following paths in F from $(3, 1)$ to $(j, 5)$ for all j are of length 4:

$$(3, 1) \to (3, 2) \to (2, 3) \to (2, 4) \to \{(1, 5), (3, 5)\};$$

$$(3, 1) \to (3, 2) \to (2, 3) \to (3, 4) \to (2, 5).$$

Thus there are paths of length not exceeding $n/2$ from $(3, 1)$ to (j, q) for all j and $5 \leqslant q \leqslant (n/2) + 1$. For $q = 3, 4$, the following paths from $(3, 1)$ to (j, q) are of length not exceeding $n/2$:

$$(3, 1) \to (3, 2) \to (1, 1) \to (2, 2) \to \{(1, 3), (3, 3)\};$$

$$(3, 1) \to (3, 2) \to (2, 3);$$

$$(3, 1) \to (3, 2) \to (1, 1) \to (2, 2) \to (1, 3) \to (1, 4);$$

$$(3, 1) \to (3, 2) \to (2, 3) \to \{(2, 4), (3, 4)\}.$$

For $(n/2) + 2 \leqslant q \leqslant n - 1$, since $(3, 1) \to \{(1, n), (3, n)\}$ in F, the proof is similar to that for $(1, 1)$.

Consider $(1, 2)$ and (j, q) for $6 \leqslant q \leqslant (n/2) + 2$. Note that the following paths in F from $(1, 2)$ to $(j, 6)$ for all j are of length 4:

$$(1, 2) \to (2, 3) \to (2, 4) \to (1, 5) \to \{(1, 6), (2, 6)\};$$

$$(1, 2) \to (2, 3) \to (2, 4) \to (3, 5) \to (3, 6).$$

Thus there are paths of length not exceeding $n/2$ from $(1, 2)$ to (j, q) for all j and $6 \leqslant q \leqslant (n/2) + 2$. For $q = 4, 5$, the following paths from $(1, 2)$ to (j, q) are of length not exceeding $n/2$:

$$(1, 2) \to (2, 3) \to (2, 2) \to (1, 3) \to (1, 4);$$

$$(1, 2) \to (2, 3) \to \{(2, 4), (3, 4)\};$$

$$(1, 2) \to (2, 3) \to (2, 4) \to \{(1, 5), (3, 5)\};$$

$$(1, 2) \to (2, 3) \to (3, 4) \to (2, 5).$$

For $(n/2) + 3 \leqslant q \leqslant n$, since $(1, 2) \to (3, 1) \to \{(1, n), (3, n)\}$ and $(1, 2) \to (2, 1) \to (2, n)$ are paths of length 2 from $(1, 2)$ to (j, n) for all j, there are paths of length not exceeding $n/2$ from $(1, 2)$ to (j, q) for all j and $(n/2) + 3 \leqslant q \leqslant n$.

Consider $(2, 2)$ and $4 \leqslant q \leqslant (n/2) + 2$. Since $(2, 2) \to (1, 3) \to \{(1, 4), (2, 4)\}$ and $(2, 2) \to (3, 3) \to (3, 4)$ are paths of length 2 from $(2, 2)$ to $(j, 4)$ for all j, there are paths of length not exceeding $n/2$ from $(2, 2)$ to (j, q) for all j and $4 \leqslant q \leqslant (n/2) + 2$.

For $(n/2) + 3 \leqslant q \leqslant n - 1$, note that the following paths in F from $(2, 2)$ to $(j, n - 1)$ for all j are of length 3:

$$(2, 2) \to (3, 1) \to (3, n) \to (1, n - 1);$$

$$(2, 2) \to (3, 1) \to (1, n) \to \{(2, n - 1), (3, n - 1)\}.$$

Thus there are paths of length not exceeding $n/2$ from $(2, 2)$ to (j, q) for all j and $n/2 + 3 \leqslant q \leqslant n - 1$. For $q = n$, the following paths from $(2, 2)$ to (j, q) are of length not exceeding $n/2$:

$$(2, 2) \to (3, 1) \to \{(1, n), (3, n)\};$$

$$(2, 2) \to (3, 1) \to (3, n) \to (2, 1) \to (2, n).$$

Consider (3, 2) and $4 \leqslant q \leqslant (n/2) + 2$. Since $(3, 2) \to (2, 3)$ in F, the proof is similar to that for (1, 2).

For $(n/2) + 3 \leqslant q \leqslant n - 1$, note that the following paths in F from (3, 2) to $(j, n-1)$ for all j are of length 3:

$$(3, 2) \to (1, 1) \to (3, n) \to (1, n-1);$$

$$(3, 2) \to (1, 1) \to (1, n) \to \{(2, n-1), (3, n-1)\}.$$

Thus there are paths of length not exceeding $n/2$ from (3, 2) to (j, q) for all j and $(n/2) + 3 \leqslant q \leqslant n-1$. For $q = n$, the following paths from (3, 2) to (j, q) are of length not exceeding $n/2$:

$$(3, 2) \to (1, 1) \to \{(1, n), (3, n)\};$$

$$(3, 2) \to (1, 1) \to (1, n) \to (2, 1) \to (2, n).$$

The proof of Proposition 1 is thus complete. □

Proposition 2. *When $n \geqslant 11$ and n is odd, then $C_n^{(3)} \in \mathscr{C}_0$.*

Proof. Let $F \in \mathscr{D}(C_n^{(3)})$ be defined as follows. For $1 \leqslant i \leqslant n-2$,

(i) if i is odd, $(1, i) \to (2, i+1)$, $(2, i) \to \{(1, i+1), (3, i+1)\}$, $(3, i) \to (2, i+1)$;
(ii) if i is even, $(1, i) \to \{(1, i+1), (2, i+1)\}$, $(2, i) \to \{(2, i+1), (3, i+1)\}$, $(3, i) \to (3, i+1)$;
(iii) $\{(1, 1), (2, 1)\} \to (1, n) \to (3, 1)$, $(1, 1) \to (2, n) \to \{(2, 1), (3, 1)\}$, $(3, 1) \to (3, n) \to \{(1, 1), (2, 1)\}$;
(iv) $(3, n-1) \to (1, n) \to \{(1, n-1), (2, n-1)\}$, $\{(2, n-1), (3, n-1)\} \to (2, n) \to (1, n-1)$, $\{(1, n-1), (2, n-1)\} \to (3, n) \to (3, n-1)$;
(v) for all $(s, t), (p, q)$, $1 \leqslant s,\ p \leqslant 3$, $1 \leqslant p, q \leqslant n$, if $(s, t) \nrightarrow (p, q)$ by (i)–(iv) above, then let $(p, q) \to (s, t)$.

It can be checked easily that if $p = q$, then $d_F((i, p), (j, q)) \leqslant 5$ for all $i, j = 1, 2, 3$. Now let $n' = n - 1$ and denote the orientation of $C_{n'}^{(3)}$ described in Proposition 1 by H. If H' is the subdigraph of H defined by

$$H' = H - \{(1, 1)(1, 2), (1, 1)(2, 2), (2, 1)(2, 2), (2, 1)(3, 2), (3, 1)(3, 2), (1, 2)(2, 1), (1, 2)(3, 1), (2, 2)(3, 1), (3, 2)(1, 1)\}$$

and F' is the subdigraph of F induced by $\{(i, p) | 1 \leqslant i \leqslant 3, 1 \leqslant p \leqslant n-1\}$, then H' and F' are isomorphic. This observation justifies the following:

Observation 1. If $|p - q| \leqslant (n-1)/2$, then $d_F((i, p), (j, q)) \leqslant (n-1)/2 = \lfloor n/2 \rfloor$ for $i, j = 1, 2, 3$.

Observation 1, together with the next eight observations, constitutes the proof of the inequality $d(F) \leqslant \lfloor n/2 \rfloor$.

Observation 2. If $(n+3)/2 \leqslant q \leqslant n-3$, then $d_F((i,1),(j,q)) \leqslant n-q+1$ for $i, j=1,2,3$.

Proof. Since $(n+3)/2 \leqslant q \leqslant n-3$, we have $2 \leqslant (n-1)-q \leqslant (n-5)/2$. By the proof of Proposition 1, for all (j,q), $j=1,2,3$, there exists a path of length $(n-1)-q$ from $(1,n-1)$ to (j,q). Likewise, there exists a path of length $(n-1)-q$ from $(3,n-1)$ to (j,q). Since $\{(1,1),(2,1)\} \to (1,n) \to (1,n-1)$ and $(3,1) \to (3,n) \to (3,n-1)$ in F, there exist paths of length $n-1-q+2=n-q+1$ from $(i,1)$ to (j,q) for $i, j=1,2,3$. □

Observation 3. If $n-2 \leqslant q \leqslant n$, then $d_F((i,1),(j,q)) \leqslant 4$ for $i=1,2$ and $j=1,2,3$, and $d_F((3,1),(j,q)) \leqslant 5$ for $j=1,2,3$.

Proof. The following paths in F justify the observation:

$(1,1) \to \{(1,n),(2,n)\}$, $(1,1) \to (1,n) \to (3,1) \to (3,n)$,
$(1,1) \to (1,n) \to \{(1,n-1),(2,n-1)\}$,
$(1,1) \to (1,n) \to (2,n-1) \to (3,n) \to (3,n-1)$,
$(1,1) \to (1,n) \to (1,n-1) \to \{(1,n-2),(3,n-2)\}$,
$(1,1) \to (1,n) \to (2,n-1) \to (2,n-2)$,
$(2,1) \to (1,n)$, $(2,1) \to (1,n) \to (2,n-1) \to \{(2,n),(3,n)\}$,
$(2,1) \to (1,n) \to \{(1,n-1),(2,n-1)\}$,
$(2,1) \to (1,n) \to (2,n-1) \to (3,n) \to (3,n-1)$,
$(2,1) \to (1,n) \to (1,n-1) \to \{(1,n-2),(3,n-2)\}$,
$(2,1) \to (1,n) \to (2,n-1) \to (2,n-2)$,
$(3,1) \to (3,n)$, $(3,1) \to (3,n) \to (1,1) \to \{(1,n),(2,n)\}$,
$(3,1) \to (3,n) \to (3,n-1)$,
$(3,1) \to (3,n) \to (3,n-1) \to (1,n) \to \{(1,n-1),(2,n-1)\}$,
$(3,1) \to (3,n) \to (3,n-1) \to \{(1,n-2),(3,n-2)\}$,
$(3,1) \to (3,n) \to (3,n-1) \to (3,n-2) \to (2,n-1) \to (2,n-2)$. □

Observation 4. If $1 \leqslant p \leqslant (n-3)/2$ and $(n+3)/2 \leqslant q \leqslant n$ such that $q-p > (n-1)/2$, then $d_F((i,p),(j,q)) \leqslant (n-1)/2$ for $i, j=1,2,3$.

Proof. We first assume that $(n+3)/2 \leqslant q \leqslant n-3$. Consider any $(i,p) \in V(F)$. Clearly, there exists a path of length $p-1$ from (i,p) to some vertex $(k,1)$. By Observation 2, there exists a path of length at most $n-q+1$ from $(k,1)$ to (j,q). Thus there exists a path of length $(p-1)+(n-q+1)=n-(q-p)(<(n+1)/2)$ from (i,p) to (j,q).

Now assume that $n-2 \leqslant q \leqslant n$. We consider $p=1,2$ and $p \geqslant 3$ separately. When $p=1,2$, since $\{(1,2),(3,2)\} \to (1,1)$ and $(2,2) \to (2,1)$, by Observation 3, it is clear that $d_F((i,p),(j,q)) \leqslant 5$. When $3 \leqslant p \leqslant (n-5)/2$, then there exist a path of length $p-1$ from (i,p) to $(1,1)$ and also a path of length $p-1$ from (i,p) to $(3,1)$. From the proof of Observation 3, we see that

$$d_F((1,1),(j,q)) \leqslant 3 \ \text{ if } (j,q) \neq (3,n-1) \quad \text{and} \quad d_F((3,1),(3,n-1))=2.$$

Thus there exists a path of length at most $p-1+3=p+2(\leqslant (n-1)/2)$ from (i,p) to (j,q). When $p=(n-3)/2$, we only need to consider $q=n-1,n$. From the proof of

Observation 3, we have

$$d_F((1,1),(j,q))\leqslant 2 \ \text{ if } j=1,2 \quad \text{and} \quad d_F((3,1),(3,q))\leqslant 2.$$

Thus there exists a path of length at most $p-1+2=p+1=(n-1)/2$ from (i,p) to (j,q). $\square$

Observation 5. If $3\leqslant q\leqslant (n-3)/2$, then $d_F((i,n-1),(j,q))\leqslant q+1$ for $i,j=1,2,3$.

Proof. Since $3\leqslant q\leqslant (n-3)/2$, by the proof of Proposition 1, for all (j,q), $j=1,2,3$, there exists a path of length $q-1$ from $(2,1)$ to (j,q). Since $\{(1,n-1),(2,n-1)\}\to(3,n)\to(2,1)$ and $(3,n-1)\to(2,n)\to(2,1)$ in F, there exist paths of length $(q-1)+2=q+1$ from $(i,n-1)$ to (j,q) for $i,j=1,2,3$. $\square$

Observation 6. If $1\leqslant q\leqslant 2$, then $d_F((i,n-1),(j,q))\leqslant 4$ for $i=1,3$ and $j=1,2,3$, and $d_F((2,n-1),(j,q))\leqslant 3$ for $j=1,2,3$.

Proof. The following paths in F justify the observation:
$(1,n-1)\to(3,n)\to\{(1,1),(2,1)\}$,
$(1,n-1)\to(3,n)\to(3,n-1)\to(1,n)\to(3,1)$,
$(1,n-1)\to(3,n)\to(1,1)\to(2,2)$,
$(1,n-1)\to(3,n)\to(2,1)\to\{(1,2),(3,2)\}$,
$(2,n-1)\to(2,n)\to\{(2,1),(3,1)\}$,
$(2,n-1)\to(3,n)\to(1,1)$,
$(2,n-1)\to(2,n)\to(2,1)\to\{(1,2),(3,2)\}$,
$(2,n-1)\to(2,n)\to(3,1)\to(2,2)$,
$(3,n-1)\to(2,n)\to\{(2,1),(3,1)\}$,
$(3,n-1)\to(2,n)\to(1,n-1)\to(3,n)\to(1,1)$,
$(3,n-1)\to(2,n)\to(2,1)\to\{(1,2),(3,2)\}$,
$(3,n-1)\to(2,n)\to(3,1)\to(2,2)$. $\square$

Observation 7. If $(n+3)/2\leqslant p\leqslant n-1$ and $1\leqslant q\leqslant (n-3)/2$ such that $p-q>(n-1)/2$, then $d_F((i,p),(j,q))\leqslant (n-1)/2$ for $i,j=1,2,3$.

Proof. We first assume that $3\leqslant q\leqslant (n-3)/2$. Consider any $(i,p)\in V(F)$. Clearly, there exists a path of length $n-1-p$ from (i,p) to some vertex $(k,n-1)$. By Observation 5, there is a path of length at most $q+1$ from $(k,n-1)$ to (j,q). Thus there exists a path of length $(n-1-p)+q+1=n-(p-q)(<(n+1)/2)$ from (i,p) to (j,q).

Assume now that $1\leqslant q\leqslant 2$. We consider $p=n-2,n-1$ and $(n+3)/2\leqslant p\leqslant n-3$ separately. When $p=n-2,n-1$, since $\{(1,n-2),(3,n-2)\}\to(2,n-1)$ and $(2,n-2)\to(1,n-1)$ in F, by Observation 6, it is clear that $d_F((i,p),(j,q))\leqslant 5$. When $(n+5)/2\leqslant p\leqslant n-3$, there exists a path of length $(n-1-p)$ from (i,p) to $(2,n-1)$. From Observation 6, we see that there exists a path of length not exceeding $n-1-p+3=n-p+2(\leqslant(n-1)/2)$ from (i,p) to (j,q). When $p=(n+3)/2$, we only need to consider $q=1$. From the proof of Observation 6, we see that $d_F((2,n-1),(j,1))=2$

for all $j=1,2,3$. Since there is a path of length $n-1-p$ from $(i,\ p)$ to $(2,n-1)$, there exists a path of length $n-1-p+2=n-p+1=(n-1)/2$ from $(i,\ p)$ to $(j,1)$. $\square$

Observation 8. For all $i,\ j=1,2,3$, $d_F((i,\ (n-1)/2),\ (j,n))\leqslant(n-1)/2$ and $d_F((i,\ (n+1)/2),\ (j,n))\leqslant(n-1)/2$.

Proof. Clearly, there are paths of length $(n-1)/2-1$ from $(i,\ (n-1)/2)$ to $(1,1)$ and $(3,1)$. Since $(1,1)\rightarrow\{(1,n),(2,n)\}$ and $(3,1)\rightarrow(3,n)$, there exists a path of length $(n-1)/2-1+1=(n-1)/2$ from $(i,\ (n-1)/2)$ to (j,n). Likewise, there are paths of length $(n-1)-((n+1)/2)$ from $(i,\ (n+1)/2)$ to $(2,n-1)$ and $(3,n-1)$. Since $(2,n-1)\rightarrow\{(2,n),(3,n)\}$ and $(3,n-1)\rightarrow(1,n)$, there exists a path of length $(n-3)/2+1=(n-1)/2$ from $(i,\ (n+1)/2)$ to (j,n). $\square$

Observation 9. For all $i,j=1,2,3$ and $q=1,2,\ldots,n-1$, $d_F((i,n),(j,q))\leqslant(n-1)/2$.

Proof. It is clear that $d_F((i,n),(j,1))\leqslant3$ and $d_F((i,n),(j,n-1))\leqslant3$ for all $j=1,2,3$. Consider $2\leqslant q\leqslant(n-1)/2$. Note that $(2,1)\rightarrow\{(1,2),(3,2)\}$ and $(3,1)\rightarrow(2,2)$. Since $(2,n)\rightarrow\{(2,1),(3,1)\}$, there exists a path of length 2 from $(2,n)$ to $(j,2)$ and consequently, there exists a path of length q from $(2,n)$ to (j,q). Similarly, note that $(2,1)\rightarrow\{(1,2),(3,2)\}$ and $(1,1)\rightarrow(2,2)$. Since $(3,n)\rightarrow\{(1,1),(2,1)\}$, similar arguments show that there exists a path of length q from $(3,n)$ to (j,q).

Consider $(1,n)$. Note that $(1,n)\rightarrow(3,1)\rightarrow(2,2)\rightarrow(2,3)\rightarrow\{(1,4),(3,4)\}$ and $(1,n)\rightarrow(3,1)\rightarrow(2,2)\rightarrow(3,3)\rightarrow(2,4)$. Thus there exists a path of length 4 from $(1,n)$ to $(j,4)$ and consequently, there exists a path of length q from $(1,n)$ to (j,q), where $4\leqslant q\leqslant(n-1)/2$. For $q=2,3$, the following paths of length not exceeding 5 justify the observation:

$(1,n)\rightarrow(3,1)\rightarrow(2,2)\rightarrow(2,1)\rightarrow\{(1,2),(3,2)\}$,
$(1,n)\rightarrow(3,1)\rightarrow(2,2)\rightarrow\{(2,3),(3,3)\}$,
$(1,n)\rightarrow(3,1)\rightarrow(2,2)\rightarrow(3,3)\rightarrow(1,2)\rightarrow(1,3)$.

Consider $(n+1)/2\leqslant q\leqslant n-2$. Note that $(1,n-1)\rightarrow\{(1,n-2),(3,n-2)\}$ and $(2,n-1)\rightarrow(2,n-2)$. Since $(1,n)\rightarrow\{(1,n-1),(2,n-1)\}$, there exists a path of length 2 from $(1,n)$ to $(j,n-2)$ and consequently, there is a path of length $n-q(\leqslant(n-1)/2)$ from $(1,n)$ to (j,q).

Consider $(2,n)$. Note that $(2,n)\rightarrow(1,n-1)\rightarrow(1,n-2)\rightarrow\{(2,n-3),(3,n-3)\}$ and $(2,n)\rightarrow(1,n-1)\rightarrow(3,n-2)\rightarrow(1,n-3)$. Thus there exists a path of length 3 from $(2,n)$ to $(j,n-3)$ and consequently, there exists a path of length $n-q(\leqslant(n-1)/2)$ from $(2,n)$ to (j,q), where $(n+1)/2\leqslant q\leqslant n-3$. For $q=n-2$, the following paths of length not exceeding 5 justify the observation:

$(2,n)\rightarrow(1,n-1)\rightarrow\{(1,n-2),(3,n-2)\}$,
$(2,n)\rightarrow(1,n-1)\rightarrow(1,n-2)\rightarrow(2,n-1)\rightarrow(2,n-2)$.

Consider $(3,n)$. Note that $(3,n)\rightarrow(3,n-1)\rightarrow(3,n-2)\rightarrow(1,n-3)$ and $(3,n)\rightarrow(3,n-1)\rightarrow(1,n-2)\rightarrow\{(2,n-3),(3,n-3)\}$. Thus there exists a path of length 3 from $(3,n)$ to $(j,n-3)$ and consequently, there exists a path of length $n-q(\leqslant(n-1)/2)$

from $(3, n)$ to (j, q), where $(n + 1)/2 \leqslant q \leqslant n - 3$. For $q = n - 2$, the following paths of length not exceeding 5 justify the observation:

$(3, n) \to (3, n - 1) \to \{(1, n - 2), (3, n - 2)\}$,

$(3, n) \to (3, n - 1) \to (3, n - 2) \to (2, n - 1) \to (2, n - 2)$. $\square$

The above nine observations on the orientation F justify our claim that $d(F) \leqslant \lfloor \frac{n}{2} \rfloor$ and the proof of Proposition 2 is thus complete. $\square$

It is easily checked that in the orientations F provided in Propositions 1 and 2, every vertex of $V(F)$ lies on a 4-cycle. By Propositions 1 and 2, we know that $C_n^{(3)}$ admits an orientation F with $d(F) = \lceil n/2 \rceil$ for each $n \geqslant 10$. Since $\lceil n/2 \rceil > 4$ when $n \geqslant 10$, Lemma D implies $\overrightarrow{d}(C_n(s_1, \ldots, s_n)) \leqslant \lceil n/2 \rceil$ for all $n \geqslant 10$ and $s_i \geqslant 3$ for each $i = 1, \ldots, n$. By Theorem A, $\lceil n/2 \rceil$ is also a lower bound for $\overrightarrow{d}(C_n(s_1, \ldots, s_n))$, Theorem (a) now follows.

4. Proofs of parts (b) and (c) in the main theorem

Proposition 3. $C_n^{(3)} \in \mathscr{C}_1$ *for* $n = 6, 7$.

Proof. Note that $d(C_6) = d(C_7) = 3$. We shall prove the result for $C_6^{(3)}$. The result for $C_7^{(3)}$ can be shown similarly. By Theorem C, we only need to show that if $F \in \mathscr{D}(C_6^{(3)})$, then $d(F) > 3$.

Let $F \in \mathscr{D}(C_6^{(3)})$. First observe that for all $(i, j) \in V(C_6^{(3)})$, if $(i, j) \to \{(k, j + 1) | k = 1, 2, 3\}$ in F, then $d_F((1, j + 2), (i, j)) \geqslant 4$. Consider the vertex $(1, 1)$. Since $C_6^{(3)}$ is a 6-regular graph, we may assume, without loss of generality, that $|\mathrm{O}((1, 1))| \leqslant 3$ and $\{(2, 2), (3, 2)\} \to (1, 1) \to (1, 2)$ in F. Now if $d(F) \leqslant 3$, then $d_F((1, 1), (i, 3)) \leqslant 3$ for $i = 1, 2, 3$ which implies that $(1, 2) \to \{(i, 3) | i = 1, 2, 3\}$. But then $d(F) \geqslant 4$ by the above observation. Thus $C_6^{(3)} \in \mathscr{C}_1$. $\square$

Proposition 4. $C_n^{(3)} \in \mathscr{C}_1$ *for* $n = 8, 9$.

Proof. Note that $d(C_8) = d(C_9) = 4$. We shall prove the result for $C_8^{(3)}$. The result for $C_9^{(3)}$ can be shown similarly. Again, we only need to show that if $F \in \mathscr{D}(C_8^{(3)})$, then $d(F) > 4$.

Let $F \in \mathscr{D}(C_8^{(3)})$ and suppose $d(F) \leqslant 4$. For each $(p, q) \in V(F)$, define

$$A(p, q) = \{(i, q + 1) \,|\, 1 \leqslant i \leqslant 3, (p, q) \to (i, q + 1) \text{ in } F\},$$

$$B(p, q) = \{(j, q - 1) \,|\, 1 \leqslant j \leqslant 3, (p, q) \to (j, q - 1) \text{ in } F\}.$$

Similar to the proof of Proposition 1, if we assume $d(F) \leqslant 4$, then $|A(p, q)| \neq 3$ and $|B(p, q)| \neq 3$ for all $(p, q) \in V(C_8^{(3)})$. By considering the converse of F, it is easy to see that if $d(F) \leqslant 4$, then $|A(p, q)|$ and $|B(p, q)|$ are both non-zero. We shall now consider two cases.

Case 1. For some $1 \leqslant q \leqslant 8$, say $q = 1$, we have $|A(i, 1)| = 2$ for all $i = 1, 2, 3$.

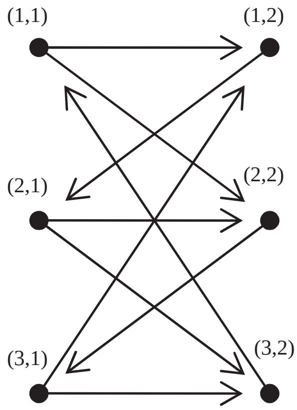

Fig. 1.

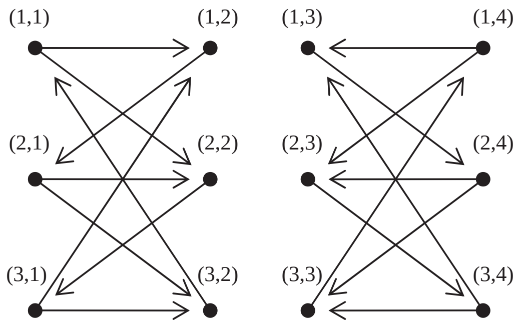

Fig. 2.

Since $|A(i,1)| = 2$ and $|B(i,2)| \neq 0$ for all $i = 1, 2, 3$, we may assume, without loss of generality, that $(3,2) \to (1,1) \to \{(1,2),(2,2)\}$, $(1,2) \to (2,1) \to \{(2,2),(3,2)\}$ and $(2,2) \to (3,1) \to \{(1,2),(3,2)\}$ in F. (See Fig. 1 .)

Since $d(F) \leqslant 4$, $d_F((i,4),(j,1)) = 3$ for all $i, j = 1, 2, 3$. Now if $|B(1,4)| = 1$, then $d_F((1,4),(j,1)) = 3$ implying $|B(i,3)| = 3$ for some i, which is not possible. Thus $|B(i,4)| = 2$ for all $i = 1, 2, 3$. We may now assume, without loss of generality, that $\{(1,4),(3,4)\} \to (1,3) \to (2,4)$, $\{(1,4),(2,4)\} \to (2,3) \to (3,4)$ and $\{(2,4),(3,4)\} \to (3,3) \to (1,4)$ in F. (See Fig. 2.)

Suppose $(1,3) \to (1,2)$ in F. (The case when $(1,2) \to (1,3)$ is similar by symmetry.) If $|B(1,3)| = 1$, then $d_F((1,4),(1,1)) = d_F((1,4),(3,1)) = 3$ implying $(2,3) \to \{(2,2),(3,2)\}$. But then $d_F((2,1),(3,4)) > 4$, a contradiction. So $|B(1,3)| = 2$. If $(3,2) \to (1,3) \to (2,2)$, then $d_F((1,1),(2,4)) > 4$; whereas if $(2,2) \to (1,3) \to (3,2)$, then $d_F((3,1),(2,4)) > 4$, a contradiction in either case.

Case 2. For all $1 \leqslant q \leqslant 8$, there exists some $1 \leqslant i_q \leqslant 3$ such that $|A(i_q, q)| = 1$.

Note that in this case, we may also assume that for all $1 \leqslant q \leqslant 8$, there exists some $1 \leqslant j_q \leqslant 3$ such that $|B(j_q, q)| = 1$. We shall assume that $\{(1,2),(3,2)\} \to (2,1) \to (2,2)$

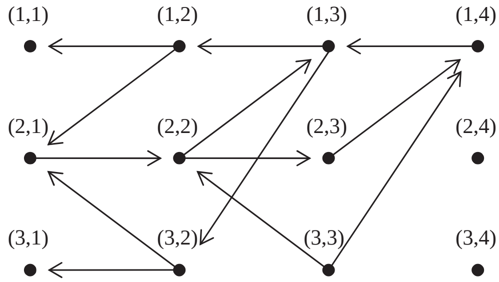

Fig. 3.

and $\{(2, 3), (3, 3)\} \to (1, 4) \to (1, 3)$ in F. Note that $d_F((2, 1), (i, 4)) = 3$ and $d_F((1, 4), (i, 1)) = 3$ for all $i = 1, 2, 3$. It is clear that $|B(1, 3)| = 2$ and we may assume, by symmetry, that $(1, 3) \to (1, 2)$. Now if $(3, 2) \to (1, 3) \to (2, 2)$, then $|A(2, 2)| = 2$ which implies that $(2, 2) \to \{(2, 3), (3, 3)\}$. This orientation is then similar to the case when $(2, 2) \to (1, 3) \to (3, 2)$ by considering its converse. So we may assume, without loss of generality, that $(2, 2) \to (1, 3) \to (3, 2)$ in F.

Since $d_F((2, 1), (1, 4)) = 3$, we must have either $(2, 2) \to (2, 3)$ or $(2, 2) \to (3, 3)$. Without loss of generality, we assume that $(3, 3) \to (2, 2) \to (2, 3)$ in F.

Since $d_F((1, 4), (1, 1)) = d_F((1, 4), (3, 1)) = 3$, we must have either '$(1, 2) \to (1, 1)$ and $(3, 2) \to (3, 1)$' or '$(3, 2) \to (1, 1)$ and $(1, 2) \to (3, 1)$' in F. By symmetry, we only need to consider the case that $(1, 2) \to (1, 1)$ and $(3, 2) \to (3, 1)$. (See Fig. 3.) Observe that $(3, 1) \to (1, 2)$ and $(1, 1) \to (3, 2)$. Now as $|B(1, 2)| = |B(3, 2)| = 2$, we must have $|B(2, 2)| = 1$ and thus we assume, by symmetry, that $(3, 1) \to (2, 2) \to (1, 1)$. As $|A(1, 1)| = 1$, $|A(3, 2)| = 2$. Thus $(3, 2) \to \{(2, 3), (3, 3)\}$, which in turn implies that $(2, 3) \to (1, 2) \to (3, 3)$ in F.

We now consider $|B(2, 4)|$. If $|B(2, 4)| = 2$ and $(3, 3) \to (2, 4) \to \{(1, 3), (2, 3)\}$, then $d_F((2, 1), (2, 4)) > 4$. If $(2, 3) \to (2, 4) \to \{(1, 3), (3, 3)\}$, then $(1, 3) \to (3, 4) \to (2, 3)$ but $d_F((3, 4), (3, 1)) > 4$. If $(1, 3) \to (2, 4) \to \{(2, 3), (3, 3)\}$, then $d_F((1, 1), (2, 4)) > 4$. If $|B(2, 4)|=1$ and $\{(2, 3), (3, 3)\} \to (2, 4) \to (1, 3)$, then $(1, 3) \to (3, 4) \to \{(2, 3), (3, 3)\}$ but $d_F((1, 1), (3, 4)) > 4$. If $\{(1, 3), (3, 3)\} \to (2, 4) \to (2, 3)$, then $d_F((2, 4), (3, 1)) > 4$. If $\{(1, 3), (2, 3)\} \to (2, 4) \to (3, 3)$, then $d_F((2, 4), (2, 1)) > 4$.

We have exhausted all possibilities and thus proved that if $F \in \mathscr{D}(C_8^{(3)})$, then $d(F) > 4 = d(C_8)$. The proof for $C_9^{(3)}$ is similar.

We arrive at Theorem (b) by combining Propositions 3 and 4. $\square$

Proposition 5. *$C_n^{(4)} \in \mathscr{C}_0$ for $n = 6, 7$.*

Proof. We provide two orientations for $C_6^{(4)}$ and $C_7^{(4)}$ with orientation number equal to 3. Define the following orientation F of $C_6^{(4)}$. For all $(i, j) \in V(C_6^{(4)})$,

(i) if j is odd, $(1, j) \to \{(1, j + 1), (2, j + 1)\}$, $(2, j) \to \{(3, j + 1), (4, j + 1)\}$, $(3, j) \to \{(1, j + 1), (3, j + 1)\}$ and $(4, j) \to \{(2, j + 1), (4, j + 1)\}$;

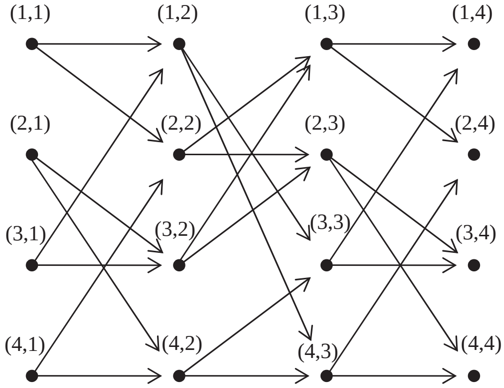

Fig. 4.

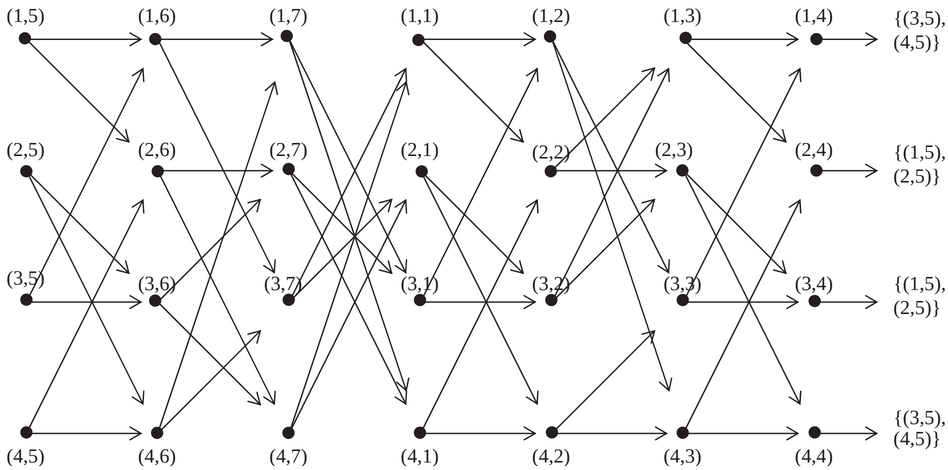

Fig. 5.

(ii) if $j = 2, 4$, $\{(1, j), (4, j)\} \to \{(3, j + 1), (4, j + 1)\}$ and $\{(2, j), (3, j)\} \to \{(1, j + 1), (2, j + 1)\}$; and if $j = 6$, $\{(1, 6), (4, 6)\} \to \{(3, 1), (4, 1)\}$ and $\{(2, 6), (3, 6)\} \to \{(1, 1), (2, 1)\}$;

(iii) for all $(i, j), (p, q) \in E(C_6^{(4)})$, if $(i, j) \nrightarrow (p, q)$ in (i) and (ii), then let $(p, q) \to (i, j)$.

Fig. 4 shows part of the orientation. Note that only those arcs described in (i) and (ii) above are shown. It is easily verified that $d(F) = 3$.

To show that $C_7^{(4)} \in \mathscr{C}_0$, we provide an orientation F' of $C_7^{(4)}$ satisfying $d(F')=d(C_7)=3$. We modify F defined for $C_6^{(4)}$ slightly, and Fig. 5 shows the orientation F'. For clarity purposes, not all arcs are shown, and for all $(i, j)(p, q) \in E(C_7^{(4)})$, if $(i, j) \nrightarrow (p, q)$ in Fig. 5, then $(p, q) \to (i, j)$ in F'. It can be checked that $d(F') = 3$, and the proof of Proposition 5, i.e., Theorem (c) is complete. □

Remarks. (i) Some other results on the orientation numbers of graphs in the family $C_n(s_1, \ldots, s_n)$ were obtained in [12]. The table below summarizes those results together with what are presented in this paper. A pair $\{x, y\}$ of integers is a co-pair if

$$x \leqslant y \leqslant \binom{x}{\lfloor \frac{x}{2} \rfloor}.$$

(ii) In this paper, we are considering minimum diameter orientations of *undirected* graphs, whereas a closely related problem would be to consider an orientation of a *digraph* [7]. An orientation of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for every pair $x \neq y$ of vertices such that both xy and yx are in D. The following two theorems which, respectively, generalizes Theorems A and B in this paper were established in [7].

n		$\in \mathscr{C}_0, \mathscr{C}_1$ or $\mathscr{C}_2$?
3	$C_3^{(s)}$, $s \geqslant 2$	$\mathscr{C}_1$
4	$C_4(s_1, s_2, s_3, s_4)$, $s_i \geqslant 2$	$\mathscr{C}_1$ if $\{s_1 + s_3, s_2 + s_4\}$ is a co-pair, $\mathscr{C}_2$ otherwise
5	$C_5^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$
6	$C_6^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$ if $s = 3$, $\mathscr{C}_0$ if $s = 4$
7	$C_7^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$ if $s = 3$, $\mathscr{C}_0$ if $s = 4$
8	$C_8^{(3)}$	$\mathscr{C}_1$
	$C_8(s_1, \ldots, s_8)$, $s_i \geqslant 4$	$\mathscr{C}_0$
9	$C_9^{(3)}$	$\mathscr{C}_1$
	$C_9(s_1, \ldots, s_9)$,$s_i \geqslant 4$	$\mathscr{C}_0$
$\geqslant 10$	$C_n(s_1, \ldots, s_n)$, $s_i \geqslant 3$	$\mathscr{C}_0$

Theorem. *Let D be a strong digraph of order $n \geqslant 3$ and $s_i \geqslant 2$ for each $i = 1, 2, \ldots, n$. Then $d(D) \leqslant \vec{d}(D(s_1, \ldots, s_n)) \leqslant d(D) + 2$.*

Theorem. *If $d(D) \geqslant 4$ and $s_i \geqslant 4$ for each $i = 1, \ldots, n$, then $\vec{d}(D(s_1, \ldots, s_n)) = d(D)$.*

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